SPACE-TIME GEOMETRY OF A TORUS FOR A LORENTZ- AND CONFORMAL-IN Variant STRING THEORY WITHOUT DIVERGENCES†

A.O.E. Animalu and N.C. Animalu

Dept. of Physics and Astronomy, University of Nigeria, Nsukka, Nigeria & Int. Centre for Basic Research, Limpopo St. FHA, Maitama, Abuja, Nigeria
email: nscience@aol.com

Abstract

From an analysis based on linguistic/projective geometry of the meeting point(s) of the quantum culture(s) of the East and West and a pre-historic African (Igbo-Ukwu) culture that built a hyperdoughnut, i.e. a torus in bronze cast (c.10,000-5,000 BC), we present a strong case for ab initio use of the space-time geometry of a torus for the construction of Lorentz- and conformal-invariant string theory of everything without divergences. From this perspective, we show firstly, that the Nambu-Goto action integral associated with an element of surface area of the torus generates an Eulerian beta-function of the type envisaged in the Veneziano (dual resonance) model of hadrons, and that the conformal (non-unitary topological) mapping of the torus onto a two-sheeted (Riemann) plane provides new insight into the relation of modular functions to the theory of interacting strings. Secondly, we develop a non-phenomenological purely geometric approach to compactification of M-theory in 11-dimensions and superstring theories in 10- and 26-dimensions based on the principle of duality in projective space-time geometry of the torus which is equivalent to supersymmetry transformations between vectors and antisymmetric tensors in 4-dimensional Minkowski space. Compatibility of particle-antiparticle (boson), open-string variables (with internal SU(2) symmetry) and three-fermion close-string variables equivalent to plane coordinates (with internal SU(3) symmetry) then leads to Maxwell’s type of consistency equations and current conservation as well as a possible explanation of the vanishing of the cosmological constant. Thirdly, we express the metric tensor of the 5-dimensional superspace of a torus in terms of

anticommuting Dirac matrices and relate it to the 32x32-matrix representation of creation and annihilation operators of the group SO(2N) for N=5. Finally we outline how the present theory can be tested experimentally and draw conclusions.

1. INTRODUCTION

One of the outstanding problems of contemporary quantum physics culture is to establish a mathematically self-consistent relation between the usual mind/body problem of consciousness in the universe and the wave/particle duality problem of quantum mechanics, i.e. Heisenberg’s uncertainty principle, or in other words, the problem of unifying the descriptions of the whole universe (macrocosm) and its minutest parts (microcosm). The famous Schrodinger cat experiment in which the cat always responds according to the expectation of the observer has led to the conclusion that the cat is conscious and aware of the mind-set of the observer, and responds in consonance with that mindset as if it has its own “mind” that resides in the “wave field” associated with the motion of its parts. As recalled by Pauli (1958a, p.260) in his Writings on Physics and Philosophy, “after Kepler(1600) had shown that the optical images on the retina are inverted in relation to the original objects, he baffled the scientific world for a while by asking why then people did not see objects upside down instead of upright”. No wonder Alva and Heidi Toffler (1981) on p.46 of their book entitled The Third Wave made the following remarkable statement: “Every civilization has a hidden code – a set of rules and principles that run through all its activities like a repeated design”. The significance of this statement emerges when we classify the revolutionary epochs of modern physics into three distinct waves.

The First Wave Physics began in Renaissance Europe with the publication of Kepler’s three empirical laws of planetary motion in the year 1600, and Newton’s synthesis of these laws in the 1660s into the deterministic laws of classical mechanics that ushered in the Machine Age. The “hidden code” that led to the synthesis lay in the Judeo-Christian Linear creation axiom which found its way into Renaissance thought through “the Cartesian rectilinear grid …[which] is suitable for analysis of the world into separately existent parts (e.g. particles and fields)” as stated by D. Bohm(1980) on p.xiv of his book entitled Wholeness and the Implicate Order. That this hidden code ran through Newton’s synthesis is evident in the questions he posed at the end of his researches in Optics, following his apparently futile attempt to construct a linear and mechanical theory of light. The atrophy of the First Wave Physics, which some may regard as its triumph, began with the parting of the ways between science and religion in the wake of the trial of Galileo (c. 1613) by the Roman Church which made scientists, thereafter, to shun the twilight zone between science and religion. Yet, scientists
continue, consciously or unconsciously, to roam in the twilight zone in search of an understanding of man himself.

The Second Wave Physics began three hundred years later with the birth of quantum mechanics of atoms and molecules in 1900. Quantum mechanics still retained the rectilinear grid (extended slightly in Einstein’s theory of relativity to the curvilinear grid) in that (see, p.66 of D Bohm(1980): i) The fundamental laws of the quantum theory are to be expressed with the aid of a wave function which satisfies a linear equation (so that solutions can be superposed linearly), and ii) All physical results are to be calculated with the aid of certain ‘observables’ represented by Hermitian operators, which operate linearly on the wavefunction. But Heisenberg’s indeterminacy principle shattered the determinism of Newtonian Mechanics, while relativistic quantum mechanics revealed the inconsistency of the atomistic nature of the linear Cartesian order with the interconnectedness or wholistic order of the known families of the elementary particles, as described by Fritjof Capra(1978) in his book entitled The Tao of Physics. The achievement of the Second Wave Physics lies in the triumphs of electronics technology that has ushered in the Computer Age – a triumph of silicon over steel in the creation of the new information wealth of the developed nations. The atrophy of second Wave Physics began with the parting of ways between the atomistic and the wholistic order in the wake of the breakdown in communication/dialogue between the eminent founding fathers of the quantum theory, Bohr and Einstein, summed up in the justly famous Einstein dictum: “God does not play dice with the universe”.

The Third Wave Physics is currently under way. Its objective is to achieve a synthesis of the Cartesian atomistic order and the interconnectedness or wholistic order found in living things or the so-called self-organizing or growing systems; or as Nick Herbert (1985) puts it on p. 249 of a book entitled, Quantum Reality, Beyond the New Physics: “One of the greatest scientific achievements imaginable would be the discovery of an explicit relationship between the waveform alphabets of quantum theory and certain human states of consciousness” Since consciousness is both wholistic and atomistic, an even greater achievement may lie, not only in the quantization of human states of consciousness but in the geometrization of quantum theory in order to make it comprehensible not only to the human senses but also to the mind. For according to Plato: “what is perceptible to the senses is the reflection of what is intelligible to the mind”. In other words, instead of the Judeo-Christian order expressed by the Pauline dictum (Romans 1:20): “Per visibila ad invisibila”, which has often been interpreted in the Western World to mean that the invisible or spiritual order is subservient to the material order, Third Wave Physics allows for the dual principle: Per invisibila ad visibila, i.e. a material order subservient to the
spiritual order. It is no wonder then that Western philosophers and physicist have long been wondering how the spiritual world, so obviously real, could so completely elude detection by the increasingly sensitive instruments of science. The answer may well lie in the inadvertent obliteration of the wholistic view – like the proverbial jealous dog carrying a piece of bone in the mouth and arriving at the bank of a river. On seeing its own distorted image with bone in its mouth reflected from the river (and lacking the human consciousness of image recognition), jealously opens its mouth to bark/grab the bone from the other dog reflected in the water, and lost the bone into the depth of the water, in apparent confusion of greed and “chasing its own shadow”. Put in other words, a physicist solving Einstein’s equations of general relativity soon arrives at a singularity or a “boundary” of the universe. On looking at the distortion of his field equations by the “image “ of his universe in the boundary, the physicist abandons his deterministic equations by jumping into the sea of indeterminate “quantum fluctuations”, where he loses consciousness of the whole and talks only about the “early universe”. To regain consciousness, the Third Wave Physicist has to geometrize the quantum fluctuations by eliminating the boundary through topological folding of his Cartesian cubic grid into a torus.

It is this paradigm of Third Wave Physics espoused in Kaku’s(1995) ten-dimensional hyperspace view of the (superstring) theory of everything (TOE) (unifying the four basic (electromagnetic, weak, strong and gravitational) forces in nature and Steven Hawking’s(1986) torus view of the universe (based on his use of quantum fluctuations to eliminate the black-hole singularity of Einstein’s general relativity theory of gravitation) that we wish to recast from the African perspective described in the preceding paper by Animalu and Acholonu(2010). Our interest (see, Fig. 1 below) is to determine how the facial scarification (“ichi”) linguistic geometry of the alphabet system of the spoken (Igbo) language of the prehistoric African (Igbo-Ukwu) culture that produced an “enigmatic tyre” (i.e. a torus) in the bronze cast excavated in the 1950s by the British archaeologist, Thurston Shaw(1950) provides meeting point(s) of various (East, West and African) cultures with modern physics especially the string theory of supergravity, called the M-theory, and its further development into the superstring theory of everything (see, especially, the series of articles in Cambridge University Press published Superstrings, A Theory of Everything? edited by Davies and Brown(1988); Michio Kaku’s (1995) book entitled Hyperspace with subtitle A Scientific Odyssey Through Parallel Universes, Time Warps, and the 10th Dimension; and Time Magazine Dec. 31, 1999 edition on Albert Einstein as Person of the Century). While the M-theory is constructed in 11-dimensional hyperspace geometry inhabited by weird objects called branes, superstring theory distinguishes an open string model with gauge group SO(32) and a closed (“heterotic”) string model with gauge group E8xE8 for grand unification of
strong and electroweak forces with gravity in hyperspaces of 10- and 26-dimensions which will be elucidated by linguistic geometry.

Apart from the fact that current theories of everything have not accounted for the quantization of Kaku’s hyperspace/torus in such a depth as to explain the intriguing hexagonal grid pattern on the African (Igbo-Ukwu) bronze cast torus,
relativistic quantum field theory is further plagued by the existence of two singularities in Einstein’s special relativity theory, namely the singularities in the Lorentz transformation at $v = \pm c$ and in Einstein’s addition theorem at $vV = c^2$ for group velocity ($v$) and phase velocity ($V$) associated with the wave/particle duality problem. Taken together, these singularities imply the existence of faster-than-light particle states with velocities, $V = c^2/v \geq c$ for $v \leq c$ and violation of causality which is forbidden. To deal with these problems, we are compelled to found Einstein’s principle of relativity ab initio on the space-time geometry of a torus as we proceed to present in this paper outlined as follows.

A reformulation of Einstein’s principle of relativity on a torus will be presented in Sec. 2.2: it will have the consequence that it leads not only to a realization of the beta-function discovered “accidentally” by Veneziano(1974), but also to a theory of conformal (non-unitary) transformation of the torus onto a two-sheeted (Riemann) plane with a number of interesting physical features. Among the physical features of the conformal (non-unitary) transformation is a projective geometric method for compactifying hyperspace manifold down to the usual 4-dimensional manifold of point-particle field theories. This will be presented in Sec. 3 along with an extension of the method provided by geometric principle of duality to compactification of both 10- and 26-dimensional manifolds. In Sec. 4 we shall develop extended relativity on the torus and determine the constraints on the algebraic dimensions of a second-quantized theory from the $2^N\times 2^N (=32\times 32)$-matrix representation of creation and annihilation operators that generate the group $SO(2N)$ for $N=5$. In Sec. 5 we shall outline how the present theory could be tested and draw conclusions.

2. FORMULATION OF STRING THEORY ON A TORUS

2.1 Reformulation of Einstein’s Principle of Relativity on a Torus

In order to enable the (Cartesian) cubic frame of reference to be used in characterizing quantization of space-time geometry so as to account for the distinctive hexagonal grid on the surface of the Igbo-Ukwu bronze torus(see, Fig. 2) in ab initio formulation of Einstein’s principle of relativity, we presume that such a torus (built from equal parts of matter and antimatter, in accordance with the “big bang” theory) is defined in 3-dimensional ($x,y,z$)-space by the parametric equations (Spinger 1957, p. 34):

$$x = (s + ct \cos \theta) \cos \phi, \quad y = (s + ct \cos \theta) \sin \phi, \quad z = ct \sin \theta, \quad (2.1.0)$$
where \( t \) is the time, \( c \) the speed of light, \( s \) the proper distance, \( \phi \) the meridian angle, and \( \theta \) the latitude angle.

On eliminating \( \phi \) and \( \theta \) from (2.1.1a) we obtain an equation of the torus in the forms.

\[
s^2 + c^2 t^2 - x^2 - y^2 - z^2 = 2c ts \sin \theta
\]  \hspace{1cm} (2.1.1a)

i.e., \[
s^2 + c^2 t^2 - x^2 - y^2 - z^2 = 2s \sqrt{(c^2 t^2 - z^2)}.
\]  \hspace{1cm} (2.1.1b)

or, \[
(s^2 + c^2 t^2 - x^2 - y^2 - z^2)^2 = 4s^2 \left(c^2 t^2 - z^2\right).
\]  \hspace{1cm} (2.1.1c)

or, \[
s^4 - 2s^2 \left(c^2 t^2 + x^2 + y^2 - z^2\right) + \left(c^2 t^2 - x^2 - y^2 - z^2\right)^2 = 0.
\]  \hspace{1cm} (2.1.1d)
A generalization to ellipsoidal torus having generalized Lorentz symmetries may be incorporated via the substitutions, \( x \rightarrow b_1 x \), \( y \rightarrow b_2 y \), \( z \rightarrow b_3 z \), \( ct \rightarrow b_0 ct \), in the above equations. In the canonically conjugate energy-momentum \( p_\mu \) (\( \mu = 0,1,2,3 \)) and mass variables, the analogous equations are:

\[
p_1 = (mc + p_0 \cos \theta) \cos \varphi, \quad p_2 = (mc + p_0 \cos \theta) \sin \varphi, \quad p_3 = p_0 \sin \theta, \quad (2.1a)
\]

i.e.

\[
m^2 c^2 + p_0^2 - p_1^2 - p_2^2 - p_3^2 = 2mc\sqrt{(p_0^2 - p_3^2)}, \quad (2.1b)
\]

or

\[
(m^2 c^2 + p_0^2 - p_1^2 - p_2^2 - p_3^2)^2 = 4(mc)^2(p_0^2 - p_3^2) \quad (2.1c)
\]

or,

\[
(mc)^4 - 2(mc)^2(p_0^2 + p_1^2 + p_2^2 - p_3^2) + (p_0^2 - p_1^2 - p_2^2 - p_3^2)^2 = 0. \quad (2.1d)
\]

We now observe that Eq.(2.1b) is invariant under the usual Lorentz transformation:

\[
ct' = \gamma (ct - (v/c)z), \quad x' = x, \quad y' = y, \quad z' = \gamma (z - vt), \quad (2.4)
\]

where \( \gamma \equiv 1/\sqrt{1-v^2/c^2} \). That is, as can be verified directly,

\[
(s^2 + c^2 t'^2 - x'^2 - y'^2 - z'^2) = 4s\sqrt{(c^2 t^2 - z^2)}.
\]

Moreover, if in Eq.(2.1d) \( s \neq 0 \) but

\[
\eta^{\mu \nu} x_\mu x_\nu \equiv c^2 t^2 + x^2 + y^2 - z^2 = 0 \quad \text{with} \quad (\eta_{\mu \nu}) = \text{Diag.}(+1,+1,+1,-1) \quad (2.5a)
\]

we have the “extended” relativity relation

\[
s^2 = \pm(c^2 t^2 - x^2 - y^2 - z^2) = \pm g_{\mu \nu} x_\mu x_\nu \quad \text{with} \quad g_{\mu \nu} = \text{Diag.}(+1,-1,-1,-1) \quad (2.5b)
\]

and \( (x_0, x_1, x_2, x_3) \equiv (ct, x, y, z) \). This guarantees that Einstein’s relativity principle founded on the Lorentz invariance of Eq.(2.1b) can be used as an heuristic aid in the search for general laws of nature whose underlying geometry is the torus defined by Eqs.(2.1.0).

We also observe, by using the Dirac method of removing the square root from Eq.(2.1.2b), that one can write down a 4-component spinor wave equation

\[
(m^2 c^2 + p_0^2 - p_1^2 - p_2^2 - p_3^2)\psi = 2mc(\gamma_0 p_0 - \gamma_3 p_3)\psi. \quad (2.6)
\]

where \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu \nu} \). This is a 4-component O(4,2) wave equation which determines a more general conserved current including the so-called “convective”
currents proportional to the total momentum of the particle-antiparticle system (Animalu, 1971) than the one determined by the conventional Dirac equation.

From Eq.(2.1.1a) the quantization of the torus into a lattice results from

\[ 2(cts) \sin \theta \equiv \begin{cases} 
    \pm 2(cts), & \text{if } \theta = (n + \frac{1}{2})\pi, \\
    0, & \text{if } \theta = n\pi
\end{cases} \quad (2.1.7a) \]

i.e.,

\[ \begin{cases} 
    (ct \pm s)^2 - x^2 - y^2 - z^2 = 0, & \text{if } \theta = (n + \frac{1}{2})\pi \\
    (ct)^2 + s^2 - x^2 - y^2 = 0 \& z^2 = 0, & \text{if } \theta = n\pi
\end{cases} \quad (2.1.7b) \]

where \( n = 0, 1, 2, \ldots \) Since the only values of the angle \( \theta \) compatibles with perfect translational symmetry of a crystal lattice in three-dimensional space are those for which \( (2\cos \theta - 1) = \text{integer} \), we have \( \theta = 2\pi/n \), where \( n = 1, 2, 3, 4, 6 \) include cubic and hexagonal lattices, but not pentagonal lattices (Ziman, 1960, Animalu 1977). The visual images of the types of geometric objects represented by Eqs.(2.1.7) can be constructed by rewriting the first set of equations as follows

\[ \left(x^2 + y^2 + z^2 - (ct - s)^2\right)\left(x^2 + y^2 + z^2 - (ct + s)^2\right) = 0, \quad (2.1.8a) \]

so that, for \( \theta = (n + \frac{1}{2})\pi \), a pair of concentric spheres in \( r \)-space of radii \( ct \pm s \) define a spherical shell of thickness, \( 2s \), with one sphere circumscribing the cube and the other sphere circumscribing the hexagon as shown in Fig.3: This may be termed the “black hole” solution associated with “Big Bang”.

**Fig.3:** (i) two concentric spheres of radii \( (ct \pm s) \) defined Eq. (2.1.8a); (ii) \( (u,v) \) generators of the “string” solution (2.1.8b) analogous to (iii) Kaku’s communication “worm hole”; (iv) –(vi) Symmetric objects (tetrahedron, square & hexagonal pyramids) associated with the quantized values of \( \theta = 2\pi/n \) for \( n=3,4,6 \).
The second of the two equations in (2.1.7b) is a ruled quadric surface with real 
(u, v) line-generators in 3-dimensional projective space with homogeneous 
coordinates (ct, s, x, y) given by

\[
\begin{align*}
\begin{pmatrix}
ct + x \\
s + y \\
ct - x
\end{pmatrix}
&= \begin{pmatrix}
u_1 \\
v_2
\end{pmatrix}, \\
\begin{pmatrix}
ct + y \\
s + x \\
ct - y
\end{pmatrix}
&= \begin{pmatrix}v_1 \\
v_2
\end{pmatrix},
\end{align*}
\]

\text{if } \theta = n\pi \quad (2.1.8b)

which may be termed “string” or “worm hole” solution as shown in Fig.3(ii) & (iii).

Consequently, the geometry of the torus is a promising one for 
formulating not only “extended” relativity principle and quantization of space-
time but also the Nambu-Goto action principle in string theory to which we now 
turn.

2.2 Nambu-Goto Action Principle for a Torus

We construct the metric (E,F,G) for the 2-dimensional (\theta, \varphi)-space on the 
torus by noting that, by virtue of Eqs.(2.1.1), we have

\[
\begin{pmatrix}
dx \\
dy \\
dz
\end{pmatrix} = \begin{pmatrix}
\partial x / \partial \theta & \partial x / \partial \varphi \\
\partial y / \partial \theta & \partial y / \partial \varphi \\
\partial z / \partial \theta & \partial z / \partial \varphi
\end{pmatrix}
\begin{pmatrix}
d\theta \\
d\varphi
\end{pmatrix} \quad (2.2.1a)
\]

i.e.,

\[
dx^2 + dy^2 + dz^2 = Ed\theta^2 + 2Fd\theta d\varphi + Gd\varphi^2 \quad (2.2.1b)
\]

\[
E \equiv (\partial x / \partial \theta)^2 + (\partial y / \partial \theta)^2 + (\partial z / \partial \theta)^2 = c^2 t^2,
\]

\[
F \equiv (\partial x / \partial \theta)(\partial x / \partial \varphi) + (\partial y / \partial \theta)(\partial y / \partial \varphi) + (\partial z / \partial \theta)(\partial z / \partial \varphi) = 0,
\]

\[
G \equiv (\partial x / \partial \varphi)^2 + (\partial y / \partial \varphi)^2 + (\partial z / \partial \varphi)^2 = (s + ct \cos \theta)^2.
\]

Thus, the action defined by Nambu(1970) and Goto(1971) representing the area 
of the curved surface in (\theta, \varphi)-space is given by the surface integral,

\[
A = \iint d\theta d\varphi \sqrt{EG - F^2} = \iint d\theta d\varphi (s + ct \cos \theta)ct. \quad (2.2.2)
\]
It is apparent from this that minimizing of the action \((A)\) is tantamount to minimizing both the time \((t)\) and the proper distance \((s)\), as expected in Einstein’s general relativity equations that determine the (geodesic) path of particles in space-time. But we have, in addition, a non-trivial dependence on the latitude angle \((\theta)\), which can be viewed from the following two perspectives.

### 2.3 Beta-Function and Conformal-Invariance

Firstly, if the limits of the \(\theta\)-integration are from \(\theta = 0\) to \(\theta = \pi / 2\), then Eq.(2.2.2) generates an Eulerian beta-function defined by

\[
B(q, p) = 2 \int_0^{\pi/2} \cos^{2p-1} \theta \sin^{2q-1} \theta \, d\theta = 2 \int_0^{\pi/2} \cos \theta \, d\theta, \tag{2.3.1}
\]

where \(p = 1\) and \(q = \frac{1}{2}\). The interest of this lies in the fact that it suggests an origin of the beta-function discovered by Veneziano(1974), which has been described as “an answer looking for a question”.

Secondly, a theory of conformal mapping (i.e. a non-unitary transformation) of the torus onto a two-sheeted (Riemann) plane arises by expressing Eq.(2.2.1b) in the form:

\[
(Ed\theta^2 + 2Fd\theta d\phi + Gd\phi^2)^\frac{1}{2} = (s + ct \cos \theta)(d\sigma^2 + d\xi^2)^\frac{1}{2} \tag{2.3.2}
\]

as prescribed Springer (1957, p.34), where \(\xi = \phi\) and \(\sigma\) is given by

\[
\sigma = \int_0^\theta d\theta \, ct \ln(s + ct \cos \theta) = (ct / A) \ln \left( \frac{s + ct + A \tan \frac{1}{2} \theta}{s + ct - A \tan \frac{1}{2} \theta} \right) \tag{2.3.3}
\]

(Beyer (1987, p. 266), where \(A = (c^2t^2 - s^2)^\frac{1}{2}\). This means that \((\sigma, \xi)\) are the isothermal coordinates (Springer 1957, p.19) on the torus which maps the torus (as \(\theta\) and \(\phi\) vary between \(-\pi\) and \(+\pi\) into a rectangular region \((-\pi \leq \xi \leq \pi\) and \(-ct / A \leq \sigma \leq +ct / A\)) in the \((\sigma, \xi)\)-plane. Note that space-like regions \((s^2 < 0)\) require a complex-valued \(\sigma\).

On the light cone \((s = 0)\), Eq.(2.3.3) reduces to \(\tanh \frac{1}{2} \sigma = \tan \frac{1}{2} \theta\) which must reduce to the result obtained by computing Eq.(2.3.3) in the form

\[
\sigma = \int_0^\theta d\theta / \cos \theta = \int_0^\xi dz \, ct \ln(c^2t^2 - z^2) = \frac{1}{2} \ln \left( \frac{ct + z}{ct - z} \right) \tag{2.3.4}
\]
i.e., \( \tanh \sigma = z / ct = \sin \theta \) \hfill (2.3.5)

by virtue of Eq.(2.1.0). Thus the conformal transformation may be written

\[ W = \sigma + i \xi = \frac{1}{2} \log \left( \frac{ct + z}{ct - z} e^{2i\varphi} \right) \quad \text{or} \quad \tanh W = \sin(\theta + i \varphi') \quad (2.3.6) \]

where \( \varphi' \) is appropriately defined, so that Eq.(2.3.6) is an analytical continuation of Eq.(2.3.5) into the complex plane. Then we may re-parameterize the torus defined by Eq.(2.1.1b) in the form

\[ (ct, x, y, z) = s(\cosh \sigma, i \sinh \sigma, - \sinh \sigma, \sinh \sigma) \tag{2.3.7} \]

which agrees with Eq.(2.3.5) and determines \( \sigma \) from the cross-ratio

\[ \frac{(ct - y)(x - z)}{(ct - z)(x - y)} = -ie^{2\sigma}, \]

i.e. \( W = \sigma + i \xi = \frac{1}{2} \log \left( \frac{(ct - y)(x - z)}{(ct - z)(x - y)} e^{i(2\varphi + \gamma')} \right) \). \hfill (2.3.8)

The interest of the conformal mapping is that it leads to a number of physical characteristics of string theory. Firstly, it provides a realization of the invariance of the string action under the re-labelling of the space and time coordinates on the world sheet having the topology of the torus defined by \((\theta, \varphi)\) of Eq.(2.1.1a) and by \((s, \sigma)\) in Eq.(2.3.7). Accordingly, Eq.(2.3.2) becomes, in terms of \((s, \sigma)\):

\[ A = \iint dsd\sigma \sqrt{EG - F^2} \tag{2.3.9} \]

where the \(E, F, G\) are given by the expressions (with \( \vec{r} = (x, y, z) \))

\[ E = (c\dot{t}/\dot{s})^2 - (\vec{\partial} r / \dot{s})(\vec{\partial} r / \dot{s}) = 1, \]
\[ F = (c\dot{t}/\dot{s})(c\dot{t}/\dot{\sigma}) - (\vec{\partial} r / \dot{s})(\vec{\partial} r / \dot{\sigma}) = 0, \tag{2.3.10} \]
\[ G = (c\dot{t}/\dot{\sigma})^2 - (\vec{\partial} r / \dot{\sigma})(\vec{\partial} r / \dot{\sigma}) = s^2 \]

Thus, we obtain the following result,

\[ A = \iint sd\sigma = \frac{1}{2} s^2 \sigma, \quad \text{i.e.} \quad \sigma = 2A / s^2 \tag{2.3.11} \]

which is an expression of the isothermal coordinate \((\sigma)\) in terms of \(s\) and the area \((A)\) \([\text{cf, Goursat (1959, p. 218)}]\). By comparing Eqs.(2.3.8) and (2.3.11), we infer from the periodic (many-valued) property of the logarithmic function that when
Changes by \(2\pi(N-1)\), with \(N=0,1,2,\ldots\), being an integer, the corresponding change in \(\sigma\) is given by

\[
2\Delta A / s^2 = \Delta \sigma = 2\pi(N-1), \text{ i.e., } m^2 = (\pi / G)(N-1) \tag{2.3.12}
\]

where \(G \equiv \Delta A\) is the Newtonian constant of gravitation and \(m^2 = 1/s^2\) characterizes a mass \((m)\), in units such that \(\hbar = c = 1\). This is the mass spectrum of the first quantized string theory (Gross, et al (1985), Candelas and Horwitz (1985)) for a bosonic string in \(d = 26\) space-time dimensions. For \(N=0\), Eq.(2.3.12) gives \(m^2 = -(\pi / G)\), which represents a tachyon and for \(N=1\), it gives \(m^2 = 0\) which is a massless state usually associated with the spin-2 graviton.

### 2.4 Interacting Open and Closed Strings

Another interesting physical feature of the conformal mapping of the torus onto the \(W\)-plane defined by Eq.(2.3.6) arises from the so-called level curves of a function of a complex variable, \(f(W)\) defined (Hardy 1958, p. 484) as curves for which \(|f(W)|\), called modular function, is constant. The level curves on the torus arising from \(|\tanh W| = v/c\) in the \(W\)-plane are defined, from Eq.(2.3.6), by

\[
|\sin(\theta + i\phi')| = v/c \equiv K. \tag{2.4.1}
\]

These curves (see, Fig. 4) may be interpreted as a motion along the \(z\)-axis of the torus in which a closed string expands (as \(K\) increases) and breaks up into two open strings.

---

**Fig. 4: Level curves of Eq.(2.4.1) (after Hardy(1958, p. 484, Fig. 56). The curves marked I-VIII correspond to \(K=0.35, 0.50, 0.71, 1.00, 1.41, 2.00, 2.83, 4.00\)**
Evidently, we may look upon Eq.(2.4.1) as the result of an analytical continuation of the Euclidean beta-function defined by Eq.(2.3.1) (cf. Phillips(1972, p.111 Eq.(15)):

\[
B(p, q; \theta + i\varphi') = 2 \int_{0}^{\theta + i\varphi'} \cos \theta \, d\theta = 2\sin(\theta + i\varphi'). \quad (2.4.2)
\]

If the lower limit of integration is cut off at a non-zero value of \(\theta = \theta_0\) say, then Eq.(2.4.2) may be replaced by

\[
B(p, q; \theta + i\varphi', \theta_0) = 2[\sin(\theta + i\varphi') - \sin \theta_0] \quad (2.4.3)
\]

This leads to the level curves of \(\sin(\theta + i\varphi') - C\), \(C = \sin \theta_0\) being a positive constant, which are shown for \(C < 1\) in Fig. 5, and may be interpreted as motions in which two closed strings on the torus join (as \(K\) increases) to form a new

**Fig. 5:** Level curves of \(\sin(\theta + i\varphi') - C\) for \(C=0.5\) (after Hardy(1958, p. 484, Fig. 58). The curves marked I-VII correspond to \(K=0.29, 0.37, 0.50, 0.87, 1.50, 2.60, 4.50, 7.79\).

**Fig. 6:** Level curves of \(\sin(\theta + i\varphi') - C\) for \(C=2\) (after Hardy(1958, p. 484, Fig. 59). The curves marked I-VII correspond to \(K=0.58, 1.00, 1.73, 3.00, 5.20, 9.00, 15.59\).
(larger) string and subsequently break up into two open strings. A similar case corresponding to $C > 1$ is shown in Fig. 6. If $C = 1$ the curves are the same as those of Fig. 4, except that the origin and scale are different.

2.5. Internal SU(2) Symmetry

An alternative way of looking at the conformal mapping of the torus onto the $W$-plane in the general ($s \neq 0$) case defined by Eq.(2.3.8) is in terms of an internal SU(2) symmetry embodied in the invariance of the cross-ratio of the space-time coordinates. On putting $(ct, x, y, z) \equiv (x_0, x_1, x_2, x_3)$ and $e^{2W} = \text{constant}$, Eq.(2.16) may be rewritten in the form (with $\varphi = -\pi/4$):

\[(x_0 - x_2)(x_1 - x_3) - e^{2W}(x_0 - x_3)(x_1 - x_2) = 0 \quad (2.5.1a)\]

i.e., $(1 - e^{2W})(x_0x_1 + x_2x_3) + e^{2W}(x_0x_2 + x_1x_3) - (x_0x_3 + x_2x_1) = 0$

or, $(1 - e^{2W})(f_{01} + f_{23}) + e^{2W}(f_{02} + f_{32}) - (f_{03} + f_{12}) = 0, \quad (2.5.1b)$

where $f_{\alpha\beta} = x_\alpha - x_\beta$. Thus the $f$'s behave like the components of an antisymmetric tensor field,

\[f_{\alpha\beta} = x_\alpha e_\beta - x_\beta e_\alpha \text{ with } (e_0, e_1, e_2, e_3) = (1, 1, 1, 1), \quad (2.5.2)\]

which is invariant under the SU(2) gauge transformation defined by

\[x'_\alpha = a_{11}x_\alpha + a_{12}e_\alpha, \quad e'_\alpha = a_{21}x_\alpha + a_{22}e_\alpha, \quad (a_{11}a_{22} - a_{12}a_{21} = 1) \quad (2.5.3)\]

This is equivalent to the usual linear fractional transformation,

\[x_\alpha \rightarrow x'_\alpha = (a_{11}x_\alpha + a_{12})/(a_{21}x_\alpha + a_{22}) \quad (2.5.4)\]

which leaves the cross-ratio $(x_0 - x_2)(x_1 - x_3)/(x_0 - x_3)(x_1 - x_2)$ invariant. The $f$'s also obey the orthogonality (or transversality) law,

\[f_{01}f_{23} + f_{02}f_{31} + f_{03}f_{12} = 0, \quad (2.5.6)\]

which shows that $f_{\alpha\beta}$ can be identified with the free electromagnetic field, or more generally, a Yang-Mills field.

Two important features of the invariance of the cross-ratio of the coordinates under the linear fractional transformation (2.5.4) are worth noting. Firstly, there are 24 possible cross-ratios (Semple and Kneebone 1952, p.47) of
the pair \((x_\alpha, x_\beta)\) with respect to the pair \((x_\gamma, x_\delta)\) of coordinates defined as follows:

\[
\{x_1, x_2; x_3, x_0\} = \frac{(x_i - x_j)}{(x_k - x_l)}; \quad (2.5.5)
\]

But there are only 6 distinct cross-ratios, namely

\[
\{x_1, x_2; x_3, x_0\} = \lambda, \quad \{x_1, x_3; x_2, x_0\} = 1 - \lambda, \quad \{x_2, x_1; x_3, x_0\} = \frac{1}{\lambda}, \quad (2.5.6)
\]

\[
\{x_2, x_3; x_1, x_0\} = 1 - \frac{1}{\lambda}, \quad \{x_3, x_1; x_2, x_0\} = \frac{1}{1 - \lambda}, \quad \{x_3, x_2; x_1, x_0\} = \frac{\lambda}{\lambda - 1}
\]

The number 24 frequently pops up in connection with Ramanujan’s modular functions in dual resonance model.

Secondly, suppose that \((x'_0, x'_1, x'_2, x'_3) = (\infty, 0, G^{\frac{1}{2}}, 2\pi / m_p)\) where \(G\) is the Newtonian constant of gravitation and \(m_p\) the proton mass (in units such that \(\hbar = c = 1\)). Then

\[
e^{2W} = (x'_0 - x'_2)(x'_1 - x'_3)/(x'_0 - x'_3)(x'_1 - x'_2) \equiv 2\pi / m_p G^{\frac{1}{2}}
\]

i.e., \(W = \frac{1}{2} \log(2\pi / m_p G^{\frac{1}{2}})\). \hfill (2.5.7)

This is satisfied when \(2W = 1/3\alpha\), with \(\alpha = 1/137\) being the electromagnetic fine-structure constant, as may be verified by putting \(G^{\frac{1}{2}} = 1.6 \times 10^{-33} cm\) and \(1/m_p = 2.1 \times 10^{-14} cm\) to find

\[
1/\alpha = 3 \log(2\pi / m_p G^{\frac{1}{2}}) = 3 \log(8.2 \times 10^{19}) = 137.4 \quad (2.5.8)
\]

It follows from this that the self-mass of the proton computed in the renormalization theory of quantum electrodynamics with \(2\pi / G^{\frac{1}{2}}\) as (Planck) cut-off mass is finite:

\[
\delta m = m_p \left(3\alpha / 2\right) \log(2\pi / m_p G^{\frac{1}{2}}) \approx m_p / 2\pi = 149 \text{ MeV}. \quad (2.5.9)
\]

This is of order \(m_\pi = 140 \text{ MeV}\), which is the mass of the pion cloud surrounding the physical (renormalized) proton. Such a finiteness is an important feature of string theory.
3. GEOMETRIC APPROACH TO COMPACTIFICATION

3.1 Compactification of the D=10 String Theory

Let us now return to Eq.(2.3.8) involving a relationship between \( D = 3 \) dimensions of the coordinates \((x,y,z)\) and \( D = 2 \) dimensions of the coordinates \((\theta,\phi)\), in order to abstract a general method for compactifying an arbitrary \( D \)-dimensional space with coordinates \( x_\mu(y_\alpha) \), \((\mu = 0,1,..,D-1)\) to the \( d \)-dimensional space with coordinates \( y_\alpha \) \((\alpha = 0,1,..,d-1)\):

\[
dx_\mu = (\partial x_\mu / \partial y_\alpha)dy_\alpha, \quad \text{i.e.,} \quad dx^\nu dx_\mu = g^{\alpha\beta} dy_\alpha dy_\beta, \quad (3.1.1)
\]

where \( g^{\alpha\beta} = (\partial x_\mu / \partial y_\alpha)(\partial x_\nu / \partial y_\beta) \). For the important case of \( D=10 \) with coordinates \( x_\mu(y_\alpha) \), \((\mu = 0,1,..,9)\) and \( d = 4 \) with coordinates \( y_\alpha = x_\alpha \), \((\alpha = 0,1,..,9)\), we may take \((x_4,x_5,x_6) = (f_{01},f_{02},f_{03})\) and \((x_7,x_8,x_9) = (f_{23},f_{31},f_{12})\), where \( f_{\alpha\beta} = x_\alpha - x_\beta \) as discussed above. A more elaborate geometric approach to compactification based on geometric principle of duality and representation of conserved currents as coordinates (Sommerville 1959) is required for compactification of \( D=26 \) string theory to which we now turn.

3.2 Geometric Principle of Duality

The geometric principle of duality states that a light cone may be generated in either of two ways: either as the locus of points \( x_\mu \), \((\mu = 0,1,2,3)\) in 4-dimensional Minkowski space \((M^4)\) in which it is represented by the homogeneous algebraic relation,

\[
g^{\alpha\beta} x_\alpha x_\beta = x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0. \quad (3.2.1a)
\]

Or (dually) as the envelop of its complex of tangent lines \( X_{\alpha\beta} \) in 6-dimensional tangent space \((B^6)\) in which it is represented by the homogeneous algebraic relation (Semple and Kneebone 1952, p. 313) or Sommerville (1959, p.374):

\[
(g^{\alpha\beta} g^{\gamma\delta} - g^{\gamma\beta} g^{\delta\alpha}) X_{\alpha\beta} X_{\gamma\delta} = X_{01}^2 + X_{02}^2 + X_{03}^2 - X_{23}^2 - X_{31}^2 - X_{12}^2 = 0 \quad (3.2.1b)
\]
This principle is illustrated in Fig. 7 in which a circle is generated in either of two ways: either by the motion of points or the turning of tangent lines.

The relationship between Eqs. (3.2.1a) and (3.2.1b) is tantamount to supersymmetry (SUSY) between a vector $x_{\mu}$ and an antisymmetric tensor $X_{\alpha\beta} = -X_{\beta\alpha}$. The other SUSY transformation of interest below is between bosons and fermions, (represented respectively as quark-antiquark ($q, \bar{q}$) (open string) and three-quark ($q_1q_2q_3$) (closed-string) systems). The pair $(x_{\mu}, X_{\alpha\beta})$ defines a 10-vector space which undergoes by virtue of Eqs. (3.2.1a) and (3.2.1b) spontaneous compactification to $M^4 \times B^6$.

To relate $x_{\mu}$ and $X_{\alpha\beta}$ mathematically, we consider the usual Euler's transformation of a position vector $\mathbf{r} = (x_1, x_2, x_3)$ which is equivalent to a rotation through a definite angle $\theta$ about some definite line defined by the direction of a unit vector $\mathbf{e} = (e_1, e_2, e_3)$ in 3-dimensional space. This transformation is given by the formula (Sommerville 1959, p.39).

$$\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} \cos \theta + (\mathbf{a} \times \mathbf{r}) \sin \theta + (\mathbf{a} \cdot \mathbf{r}) \mathbf{a} (1 - \cos \theta)$$  

(3.2.2a)

or, in infinitesimal form (first order in small $\theta$ expansion),

$$\delta \mathbf{r} = (\mathbf{a} \times \mathbf{r}) \theta, \text{ i.e., } (\delta x_1, \delta x_2, \delta x_3) = (X_{23}, X_{31}, X_{12}),$$

(3.2.2b)
where \( X_{ij} = \theta(a_i x_j - a_j x_i) \) are components of an antisymmetric tensor of second rank in 3-dimensional space. This shows that \( x_i \) has an antisymmetric tensor (SUSY) partner \( X_{ij} \) with \( 3 = (3/2)(3-1) \) independent components equal to the number of generators of the Lie algebra of \( \text{SU}(3) \); and hence \( (x_i, X_{ij}) \) form a 6-vector space which admits a spontaneous compactification to \( M^3 \times B^3 \), where \( M^3 \) is the usual 3-dimensional space of points \( (x_i) \) and \( B^3 \) is its dual 3-dimensional space of “surface tensor” \( (X_{ij}) \) defined by (Pauli 1958b, pp.30-31).

This result has a number of additional features in 4-dimensional space \( (M^4) \) of points \( x_\alpha \) and 6-dimensional space \( (B^6) \) of “surface tensor” \( X_{\alpha \beta} \) with \( 6 = (4/2)(4-1) \) independent components equal to the number of generators of the Lie algebra of the Poincare group, \( \text{SO}(4) \) or \( \text{SO}(1,3) \), such that \( (x_\alpha, X_{\alpha \beta}) \) form a 10-vector space which admits a spontaneous compactification to \( M^4 \times B^6 \). Suppose that \( x_\alpha \) is the position 4-vector of a point particle \( (\pm e) \) say) tied by a string to a point-antiparticle \( (\mp e) \) say) located at \( x_\alpha \equiv \theta a_\alpha \), \( a_\alpha \) being a unit 4-vector, so that the \( e^- e^+ \) boson (open string) system may be characterized by the surface vector,

\[
X'_{\alpha \beta} = \bar{x}_\alpha x_\beta - \bar{x}_\beta x_\alpha = -X'_{\beta \alpha} \quad (3.2.3)
\]

Then by virtue of the relation (Pauli 1958b, p. 30, Eq.(46))

\[
\epsilon^{\alpha \beta \gamma \delta} X'_{\alpha \beta} X'_{\gamma \delta} \equiv X'_{01} X'_{23} + X'_{02} X'_{31} + X'_{03} X'_{12} = 0 \quad (3.2.4)
\]

obeyed by a surface tensor in \( M^4 \), we have on making the substitutions,

\[
X_{01} = X'_{01} + X'_{23}, \quad X_{02} = X'_{02} + X'_{31}, \quad X_{03} = X'_{03} + X'_{12}, \quad X_{23} = X'_{01} - X'_{23}, \quad X_{31} = X'_{02} - X'_{31}, \quad X_{12} = X'_{03} - X'_{12}, \quad (3.2.5)
\]

the \( \text{SO}(3,3) \)-invariant relation cited in Eq.(3.2.1b):

\[
R^{\alpha \beta \gamma \delta} X'_{\alpha \beta} X'_{\gamma \delta} \equiv X^2_{01} + X^2_{02} + X^2_{03} - X^2_{23} - X^2_{31} - X^2_{12} = 0 \quad (3.2.6)
\]

where, by using Eq.(3.2.1) as a definition of the metric tensor \( g^{\alpha \beta} \),

\[
R^{\alpha \beta \gamma \delta} \equiv g^{\alpha \beta} g^{\gamma \delta} - g^{\alpha \gamma} g^{\beta \delta} \quad (3.2.7)
\]

is a (constant) curvature tensor. This implies that, in a purely geometrical theory, an \( e^- e^+ \) (open-string) boson admits a 10-vector space representation of the form
\( (x_\alpha, X_{\alpha\beta})\) where \( x_\alpha \) is the 4-vector position of \( M^4 \) in and \( X_{\alpha\beta} \) is antisymmetric tensor (SUSY) partner related to it by Eqs.(3.2.3) and (3.2.5) with the independent components of \( X_{\alpha\beta} \) defining a 6-vector space (\( B^6 \)). Moreover, the constant curvature tensor defined by Eq.(3.2.7) is the “metric tensor” of the \( B^6 \) of \( X_{\alpha\beta} \) : this explains the vanishing of the cosmological constant of the 10-vector space under compactification to \( M^4 \times B^6 \).

In like manner, a three-particle (\( q_1, q_2, q_3 \)) fermion system admits a (closed string) representation of the form \((x_\alpha, X_{\alpha\beta}, y_\alpha, Y_{\alpha\beta}, z_\alpha, Z_{\alpha\beta})\) where \( x_\alpha, y_\alpha, z_\alpha \) are the respective coordinates of \( q_1, q_2, q_3 \) in \( M^4 \) and

\[ X_{\alpha\beta} = y_\alpha z_\beta - y_\beta z_\alpha, \quad X_{\alpha\beta} = z_\alpha x_\beta - z_\beta x_\alpha, \quad X_{\alpha\beta} = x_\alpha y_\beta - x_\beta y_\alpha \] (3.2.8)

are the “surface tensors” or Plucker’s line coordinates ((Pauli 1958b, p. 31 footnote (60a)) of the strings joining the respective particles pairs, \( q_2 q_3, q_3 q_1, q_1 q_2 \), to form the closed (fermion) string in question. In order to establish the connection of this representation with the 26-dimensional tangent space envisaged in the “heterotic” closed string model (Nambu 1970, Goto 1971) we also introduce the “volume tensor” ((Pauli 1958b, p. 30) or plane-coordinates associated with the three points \( x_\alpha, y_\alpha, z_\alpha \), via the determinant,

\[
X_{\alpha\beta\gamma} = \begin{vmatrix}
x_\alpha & y_\alpha & z_\alpha \\
x_\beta & y_\beta & z_\beta \\
x_\gamma & y_\gamma & z_\gamma
\end{vmatrix}.
\] (3.2.9)

This third rank antisymmetric tensor has only 4 independent components \((X_{123}, X_{230}, X_{301}, X_{012})\) which may be denoted by \( J_\alpha, (\alpha = 0,1,2,3) \) termed the plane-coordinates of the three points in question. Accordingly, the \( q_1 q_2 q_3 \) closed-string system may be represented geometrically by the set of coordinates \((x_\alpha, X_{\alpha\beta}, Y_{\alpha\beta}, Z_{\alpha\beta}, J_\alpha)\) forming a 26-vector defining a representative point of the 26-dimensional tangent space, which has spontaneous compactification of the form, \( M^4 \times B^6 \times B^6 \times B^6 \times \tilde{M}^4 \) where \( \tilde{M}^4 \) is the space with plane coordinates (\( J_\alpha \)).

### 3.3 Compatibility Relations and Current Conservation

The most important consequences of the above mathematical compactification scheme arise from certain compatibility relations due to the
existence of SUSY transformations between \( q_1 \) (fermion) and \( q_1 \bar{q}_1 \) (boson) on one hand, and between \( q_1 \bar{q}_1 \) (boson) open-string and \( q_1 q_2 q_3 \) (fermion) closed-string systems on the other hand. In the language of the geometric principle of duality, the compatibility in question is that the line \((L_{a\beta} \equiv \bar{x}_a x_{\beta} - \bar{x}_\beta x_a)\) representing the boson \( q_1 \bar{q}_1 \) must meet the plane \((J_a)\) representing the fermion \( q_1 q_2 q_3 \) at their common (intersection) point \((x_a)\) representing the fermion \( q_1 \) in \( M^4 \), which is the case provided that the coordinates \( x_a, x_\alpha, \bar{x}_\alpha, \bar{z}_\alpha \) are linearly dependent, and hence obey the determinantal condition:

\[
0 = \begin{vmatrix}
\bar{x}_0 & \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\
x_0 & x_1 & x_2 & x_3 \\
y_0 & y_1 & y_2 & y_3 \\
z_0 & z_1 & z_2 & z_3
\end{vmatrix} \equiv \bar{x}_0 J_0 - \bar{x}_1 J_1 - \bar{x}_2 J_2 - \bar{x}_3 J_3 \quad (3.3.1)
\]

which of current conservation type \((\partial^a J_a = 0)\). This, however, implies the existence of an algebraic consistency equation of Maxwell’s type \((\partial^\beta F_{a\beta} = J_a)\) relating the open-string variables \((L_{a\beta})\) to the closed-string variables \((J_a)\) in the form (Maxwell 1959 p.69):

\[
x^\beta L_{a\beta} = J_a \quad (3.3.2)
\]

Moreover, the open-string variables \((L_{a\beta})\) are invariant under an internal SU(2) group of transformations (equivalent to gauge-invariance of the second kind of the electromagnetic field \((F_{a\beta})\))

\[
\bar{x}'_a = a_{11} \bar{x}_a + a_{12} x_a, \quad x'_a = a_{21} \bar{x}_a + a_{22} x_a
\]

where \(a_{11} a_{22} - a_{12} a_{21} = 1\), i.e., \(L'_{a\beta} = L_{a\beta}\); and the representation of the closed string variables \((J_a)\) by 3x3 determinants via Eq.(3.2.9) implies their invariance under an internal SU(3) group of transformations. Eq.(3.3.2) is the main result of this sub-section. But this cannot be the end of the compatibility relations because we have to deal with points \( M^4 \) not on the light cone to which we now turn.
4. “EXTENDED” RELATIVITY ON THE TORUS AND CONSTRAINTS ON DIMENSIONS

Since points not on the light cone arise from the quadratic nature of Eq (2.1.1c) characterizing the torus, i.e.,

\[ s^4 + 2s^2(-c^2t^2 - x^2 - y^2 + z^2) + (c^2t^2 - x^2 - y^2 - z^2)^2 = 0, \]  

we may characterize the torus as a net of three quadrics (Semple and Kneebone 1952, p. 339): namely, the quadric defined by the vanishing of the coefficient of \( s^2 \) i.e.,

\[ 0 = c^2t^2 + x^2 + y^2 - z^2 \equiv \eta^{\mu\nu} x_\mu x_\nu \]  

(4.2a)

and the pair of quadrics defined by the remaining two equal roots of \( s^4 \)

\[ s^2 = \pm(c^2t^2 - x^2 - y^2 - z^2) \equiv g^{\mu\nu} x_\mu x_\nu, \]  

(4.2b)

where

\[
\eta = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}, \quad g^+ = \pm \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}. \]

We recognize (4.2a) as a mutation of the light cone and Eq.(4.2b) as the standard algebraic form of “extended” relativity principle. Moreover, if we use Dirac matrices to characterize the metric tensors in Eq.(4.2c) as follows,

\[ g^{\mu\nu} dx_\mu dx_\nu = (dx_0')^2 - (dx_1')^2 - (dx_2')^2 - (dx_3')^2 \\
= \pm[(dx_0')^2 - (dx_1')^2 - (dx_2')^2 - (dx_3')^2] \equiv g^{\mu\nu} dx_\mu dx_\nu \]

such that the transformation of the \( g_{\mu\nu} \) into \( g'_{\mu\nu} \) obeys the relation,

\[ g'_{\mu\nu} = \pm \frac{1}{2} \left( \gamma'_{\mu} \gamma'_{\nu} + \gamma'_{\nu} \gamma'_{\mu} \right) = \pm \frac{1}{2} \left( \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} \right) \equiv g_{\mu\nu} \]  

(4.3a)

then the upper (+) sign in this relation is preserved by the transformation of the vector \( (\gamma'_{\mu}) \) into an axial-vector \( (\gamma'_{\mu} \equiv \gamma'_{\mu} \gamma_{5}) \), giving the Lie-admissible algebraic relation.
\[ \gamma_\mu \gamma_5 = U \gamma_\mu U^{-1} \] i.e., \[ \gamma_\mu \gamma_5 U = U \gamma_\mu = 0, \] \hfill (4.3b)

where \( U = \frac{1}{\sqrt{2}} (I - \gamma_5) \), \( (\gamma_5^2 = -I) \) and \( \gamma_5 \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 \).

For the geometric principle of duality to be applicable we must replace the inhomogeneous algebraic relation, \( g^{\mu \nu} x_\mu x_\nu - s^2 = 0 \) by a homogeneous one, \( g^{\mu \nu} x_\mu x_\nu + g^{44} s^2 = 0 \) so that the compatibility requirement implies the existence of equivalent representations of (4.2b) in 5-dimensional space of 5-vectors, \( x_a \) \((a = 0, 1, 2, 3, 4)\) with \( x_4 = s \), or 10-dimensional space of the \((5/2)(5-1)=10\) independent components of the antisymmetric tensor \( X_{ab} \) which is the SUSY partner of \( x_a \), or (Salam and Strathdee, 1982) 11-dimensional Kaluza-Klein supergravity space of 11 independent components of the symmetric metric tensor (Semple and Kneebone, 1952. p. 388),

\[
(g^{44}, g^{33}, g^{22}, g^{11}, g^{00}, g^{01}, g^{02}, g^{03}, g^{23}, g^{31}, g^{12})
\]

As is well known, the constraints on the algebraic dimensions of a second-quantized fermion field is determined by Fermi-Dirac statistics through the representation of the anticommutation algebra of the single fermion creation and annihilation operators, \( c^+_i \sigma \) and \( c_i \sigma \),

\[
\{ c^+_i \sigma, c^+_j \sigma \} = c^+_i \sigma c^+_j \sigma + c^+_j \sigma c^+_i \sigma = \delta_{ij} \delta_{\sigma \sigma'}, \ 
\{ c_i \sigma, c_j \sigma \} = \{ c^+_i \sigma, c^+_j \sigma \} = 0,
\]

for \( i, j = 1, \ldots, N \), \( (\sigma = \uparrow, \downarrow) \). These operators are generators of an SO(2N) group and can be represented by \( 2^N \times 2^N \)-matrices defined by Thouless (1961). The main feature of the matrices is that the matrices decompose into \( N+1 \) blocks or submatrices of dimension

\[
\binom{N}{i} \times \binom{N}{i} \text{ with } \sum_{i=1}^{N} \binom{N}{i} = 2^N \] \hfill (4.5)

where \( i = 0, 1, \ldots, N \). Consequently, the configuration is defined by the \( K^{th} \) order Pascal triangle coefficients of \( (x+1)^N \) where \( N \) varies from 1 to \( K \). **The Pascal triangle of dimensions** is shown in Fig. 8 for \( N=1,2,\ldots,8 \).
However, “fractal” dimension (Mandebrot 1983) can also be used to express the compatibility relations for the points $x_\alpha$ not on the light cone as follows. We notice that the 10-dimensional space of pair $(x_\mu, X_{\alpha\beta} \equiv x_\alpha e_\beta - x_\beta e_\alpha)$ has compactification $M^4 \times B^6$ in the form:

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = s^2;$$

$$(x_0 - x_1)(x_2 - x_3) + (x_0 - x_2)(x_3 - x_1) + (x_0 - x_3)(x_1 - x_2) \quad (4.6)$$

if $(e_0, e_1, e_2, e_3) = (1,1,1,1)$ so that $X_{\alpha\beta} = x_\alpha - x_\beta$, as in Eq.(3.5) above. Thus the second equation in Eq.(4.6) can be the split into two cross-ratios:

$$\frac{(x_0 - x_2)(x_1 - x_3)}{(x_0 - x_3)(x_1 - x_2)} = -K^2, \quad \frac{(x_0 - x_1)(x_2 - x_3)}{(x_0 - x_3)(x_2 - x_1)} = 1 + K^2 \quad (4.7)$$

where $K$ is a “dimensionless separation parameter”. Each of the two cross-ratios is invariant under the group of “fractal” (bilinear fractional) transformations (Mandebrot 1983, p. 178):

$$x_\alpha \rightarrow x'_\alpha = (a x_\alpha + b)/(c x_\alpha + d), \quad (ad - bc = 1) \quad (4.8)$$

Also the first cross-ratio in Eq.(4.7) has the rationalized form:

$$0 = (x_0 - x_2)(x_1 - x_3) + K^2(x_0 - x_3)(x_1 - x_2)$$

$$= -\frac{1}{2}(1 + K^2)(X_{01}^2 + X_{23}^2) + \frac{1}{2}K^2(X_{02}^2 + X_{31}^2) + \frac{1}{2}(X_{03}^2 + X_{12}^2) \quad (4.9)$$
which is an SO(4,2) invariant form replacing the SO(3,3) form in Eq.(3.7) above. It implies a compactification of the 10-dimensional space of the 10-vector $(x_\alpha, X_{\alpha\beta})$ to $M^4 \times B^6$ characterized by the SO(1,3)xSO(4,2) of Poincare and “fractal” symmetry transformations. Moreover, by representing points not on the light cone in the parametric form, involving an “angular distance” ($\phi$),

$$(x_0, x_1, x_2, x_3) = s(\cosh \phi, i \sinh \phi, -\sinh \phi, \sinh \phi)$$

the cross-ratio takes the form

$$\frac{(x_0 - x_2)(x_1 - x_3)}{(x_0 - x_1)(x_2 - x_3)} \equiv \frac{(\cosh \phi + \sinh \phi)(i - 1)}{(\cosh \phi - \sinh \phi)(i + 1)} \equiv -e^{2(\phi - i\pi/4)} = -K^2,$$

$$i.e., \quad \theta = \log K$$

where $\theta = \phi - i\pi/4$ is real. This takes the standard form of Mandelbrot’s relation for fractal dimension (Mandelbrot 1983, p. 37)

$$D = \log(N)/\log(1/r)$$

on setting $\theta = (1/D)\log(N)$ and $K = 1/r$ in Eq.(4.11). In other words, compactification involving points in $M^4$ not on the light cone demand the use of non-integral (“fractal”) dimensions for consistency.

5. TEST OF THEORY THROUGH OBSERVATION OF LIGHT CAUSTICS.

We turn finally to one of the problems with string theories, namely the fact that their predictions in particle physics have not been testable (Kane, 2010). To overcome this difficult in the present paper, we now proceed to outline how our foundation of the principle of relativity on a torus in velocity space can be used to predict the observation of light “caustics” in fiber optics. We begin by drawing an analogy (in Fig.10) between the three different modes/geometries of optical fibers highlighted in Nobel Prize (2009) and deep-inelastic electron-proton fusion of a spherical $p^+$ wavepacket into the hole of $e^-$ torus as in the model of the so-called Rutherford-Santilli(1978) neutron. The analogy enables us to associate the two singularities in special relativity theory from the Lorentz transformation at $v = \pm c$ and from Einstein’s theorem on addition of velocities at $vV = c^2$ with the rectilinear and wavepacket modes respectively of the three optical fiber geometries produced by different geometries indicated in Fig 9.
By using the procedure adopted by Animalu and Ekuma (2008) for defining Bjorken variable for scaling of structure functions for deep-inelastic e-p scattering, we characterize the third mode in Fig.10 by the self-corresponding points of a non-unitary transformation \((\bar{v}, \bar{V}) \rightarrow (\bar{v}', \bar{V}')\) of a rectangular hyperboloid into a torus:

\[
\begin{align*}
\bar{v} &= \pm c \\
\bar{v}V - c^2 &= 0 \\
\epsilon &= (\bar{V}^2 - \bar{v}^2) / 2\bar{v} \bar{V} \\
x &= (P^2 - q^2) / 2q \cdot P
\end{align*}
\]
\[ \mathbf{V} \cdot \mathbf{V} \rightarrow \frac{1}{2} (\mathbf{V}^2 - \mathbf{\dot{V}}^2), \quad (5.1a) \]

i.e., \( \mathbf{\ddot{V}}^2 - \mathbf{V}^2 + 2\mathbf{V} \cdot \mathbf{\ddot{V}} = 0 \quad (5.1b) \)

This is defined in terms of the parametric equations for a torus as follows:

\[
\begin{align*}
\mathbf{v}_1 &= (iu - V \sin \theta) \cos \phi, \quad \mathbf{v}_2 = (iu - V \sin \theta) \sin \phi, \quad \mathbf{v}_3 = V \cos \theta; \\
\Rightarrow \mathbf{v}_1^2 + \mathbf{v}_2^2 + \mathbf{v}_3^2 &= V^2 - u^2 - 2iVu \sin \theta; \\
i.e., \quad 0 &= \mathbf{\ddot{v}}^2 - \mathbf{\dot{V}}^2 - 2iVu \sin \theta \equiv \mathbf{\ddot{V}}^2 - \mathbf{\dot{V}}^2 - 2 \in \mathbf{\dot{V}} . \mathbf{\ddot{V}}.
\end{align*}
\]

By rewriting Eq. (5.1a) in the matrix form

\[ q^+ \mathbf{\tau}_1 q \equiv 2\mathbf{\dot{V}} . \mathbf{\ddot{V}} \equiv (\mathbf{\ddot{V}}^2 - \mathbf{\dot{V}}^2) \equiv q^{++} \mathbf{\tau}_1' q'. \quad (5.3a) \]

we see that it involves a transformation of the underlying metric

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \mathbf{\tau}_1 \rightarrow \mathbf{\tau}_1' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

such that under two (unitary and non-unitary) transformation matrices:

\[
U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}; \quad U^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = U^+; \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad T^2 = -I. \quad (5.4a)
\]

\[
U \mathbf{\tau}_1 T - T \mathbf{\tau}_1' U = 0,
\]

has a Lie-admissible algebraic structure.

Moreover, if we re-express Eq. (5.1) in the form

\[ \mathbf{\dot{V}} \cdot \mathbf{\dddot{V}} = \frac{\mathbf{\ddot{V}}^2 - \mathbf{\dot{V}}^2}{2\mathbf{V} \cdot \mathbf{\ddot{V}}}, \quad (5.5) \]

we observe that it is separable (when \( V \) and \( v \) depend on a common local time variable) into two equations for acceleration:

\[
\mathbf{\ddot{V}} = k(\mathbf{\ddot{V}}^2 - \mathbf{\dot{V}}^2) \quad \text{(Riccati’s Eqn)} \\
\mathbf{\ddot{v}} = 2k\mathbf{\dddot{V}} \cdot \mathbf{\dddot{V}}
\]

But Eq. (5.5) can be integrated exactly to find
\[ V^2 = \frac{V^2}{3} + \frac{V_0^3}{3V}, \text{ i.e., } 3vV^2 = v^3 + V_0^3 \]  

(5.7a)

which may be expressed, on putting \( v = r/t \), as a cubic equation

\[ 3r(Vt)^2 = r^3 + (V_0t)^3, \]

i.e., \( X + Yr + r^3 = 0, \) where \( (X, Y) = (V_0t)^3, -3(Vt)^2 \))

(5.8)

This states that one would see in (X,Y)-space, the envelope of the normals to a parabola representing a “cusp” catastrophe (Poston and Stewart 1978) and shown in Fig. 10a;

![Fig.10: Envelope of the normals to (a) parabola, (b) ellipse.](image)

For an ellipse, the corresponding envelope of the normals (Fig.10b), can be seen as “light caustics” generated by passing a laser beam through a grated louver glass: this is one of the patterns that is a recurrent motif in the prehistoric African (Igbo-Ukwu) bronze artefacts (Animalu 1990, Acholonu 2009).

6. DISCUSSION AND CONCLUSION

We have shown in this paper that, as envisaged by Green(1986), the space-time geometry of the torus offers a natural geometry for formulating a Lorentz-invariant string theory without divergences and we have seen that many of the rich structure of string theory emerge from the conformal mapping of the torus onto a two-sheeted (Riemann) plane available in conventional theory of Riemann surfaces. The implication is that the universe may be a torus, as envisaged in
Eq. (2.1.0), rather than a “cosmic egg” of the big bang theory. We have shown how predictions can be made with our string theory in fiber optics and indicated an analogy with deep-inelastic e-p scattering which we shall explore further in a subsequent paper. However, because current string theories including the one described in this paper have non-unitary structure, which we have indicated in this paper, it is necessary, as pointed out by Santilli (2002) to use the appropriate mathematical structures of the Lie-admissible theory in order to avoid catastrophic inconsistency.

ACKNOWLEDGEMENT

One of the authors, Late Charles N. Animalu (1966-2004), began this research with the appearance in 1985 of M.B. Green’s popular Scientific American article on Superstring Theory while he was a BS/MS student in the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA (U.S.A.). The work was suspended in 1991 to give way for his Ph.D. research project in the Department of Physics and Astronomy, University of Nigeria, on high-Tc superconductivity which was supported by an award of Senate Research Grant No. 648 of the University of Nigeria, Nsukka. After earning Ph.D. in 1992, the work was resumed in collaboration with his father, Alex Animalu, who subsequently finalized it by 2010 in the present form.

REFERENCES


Capra, F. The Tao of Physics (Fontana/Collins, 1978)


Kepler, J. (1600): Mysterium Cosmographicum


Nobel Prize (2009), Two Revolutionary Optical Technologies compiled by the Class for Physics of the Royal Swedish Academy of Sciences


Salam, A and Strathdee, J (1982), Annals Phys. 141, p. 316 .


Time Magazine Dec. 31, 1999 edition on Albert Einstein as Person of the Century)

